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Embeddings of discrete series into some induced representations of a group of G_2 type

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Introduction

Let G be a connected semisimple Lie group with finite center, K a maximal compact subgroup of G, θ the corresponding Cartan involution. We assume $\operatorname{rank} G = \operatorname{rank} K$ in order to assure the existence of the discrete series representations of G. In this case, the discrete series representations of G are considered to be fundamental and studied for a long time. Harish-Chandra's classical work [1, Theorems 13, 16] gives a parametrization of discrete series. The discrete series of G are parametrized by regular, K-integral linear forms Λ on the complexification of a compact Cartan subalgebra of $\mathfrak{g}_0 = \mathrm{Lie}(G)$.

This parameter Λ is called the Harish-Chandra parameter and we denote by π_Λ the discrete series with Harish-Chandra parameter Λ. Then $\pi_{\Lambda_1} \simeq \pi_{\Lambda_2}$ if and only if Λ_1 and Λ_2 are conjugate under the action of the compact Weyl group of G. The method in [1] was based on the theory of character representations and his parametrization is quite an abstract one and did not give concrete realizations of discrete series.

Historically speaking, the discrete series representations appered as a subrepresentaton of the regular representation of G. But these realizations are not easy to use in investigations. So, other realizations of discrete series are given by R. Hotta and R. Parthsarathy [2], W. Schmid [5] and others. Let $(\tau_\lambda, V_\lambda)$ be the lowest K-type of the disrete series π_Λ. Then we introduce a function space $C^\infty_{\tau_\lambda}(G)$, the totality of the functions with the property $f(kg) = \tau_\lambda(k) f(g)$ for $g \in G$ and $k \in K$. Schmid intoduced a K-equivariant differential operator $\mathcal{D}_\lambda$ on $C^\infty_{\tau_\lambda}(G)$ and showed in [5] that the discrete series π_Λ can be realized as the L_2-kernel of $\mathcal{D}_\lambda$. This fact is also shown by Hotta and Parthasarathy and a simplified proof was given.

Since this kind of differential operators give the realizations of discrete series, such operators also leads us to find out embeddings of discrete series into other induced representations. By means of Szegö kernel, A. W. Knapp and N. R. Wallach gave explicit realizations for discrete series as a quotient of certain principal series in [3]. Taking contragredient representations, we find out that this expression as quotient also gives embeddings of discrete series into principal series, because the contragredient representation of a discrete series is a discrete series.

For embeddings of discrete series, W. B. Silva determined, in [6], the embeddings into principal series for groups of real rank one. Her method based on the realization given by Knapp and Wallach, and closely related to our method. But the case of the groups of higher real rank, difficulties in computation prevents us form complete determination of embeddings.

Using a modification of the operator $\mathcal{D}_\lambda$ above, H. Yamashita established, in [7], a general method to find out the embeddings of discrete series into various kind of induced representations as $(\mathfrak{g}, K)$-modules, where $\mathfrak{g}$ is the complexified Lie algebra of G. In the case of embeddings into generalized principal series, his result, Theorem 2.2, says that the dimension of the space of $(\mathfrak{g}, K)$-module homomorphism can be determined through the $(\mathfrak{l}, K_L)$-module structure of the solution space of the differential equation $\mathcal{D}_\lambda f = 0$. Here, $\mathfrak{l}$ is the complexified Lie algebra of a Levi part L of a given parabolic subgroup and $K_L = K \cap L$. This method is successfully applied to the case of $SU(2,2)$ in [7, 8].

But, since there are few results on exceptional groups, the author tried to observe the case of groups of type G_2. We have determind the embeddings of discrete series into principal series for the case of a group of G_2 type in [9]. In that paper, embeddings into generalized principal series associated to maxmal parabolic subgroups are left to be determined. So, in this article, we give the embeddings into generalized principal series induced from one of the maximal parabolic subgroups of a group of type G_2.

This article consists of three sections. In the first section, we describe the structure of the Lie algebra, of a maximal compact subgroup and of parabolic subgroups for a group of type G_2. In the succeeding section, we discuss on discrete series. Their parametrization and embeddings into principal series induced from one of the maximal parabolic subgroup are described. We also explain the methods for determination of embeddings, including the definition of the operator $\mathcal{D}_\lambda$ above. In the last section,

Our main result in this article is given as follows:

Theorem *The discrete series π_Λ can be embedded into $\mathrm{Ind}_{P_1}^G(\xi \otimes 1_{N_1})$, if and only if ξ is one of the following representations.*

If Λ is Δ_I^+-dominant, then $\xi \simeq \sigma_{r,\varepsilon}^{(1)} \otimes \chi_{s-1}$, $\overline{\pi}_{a_\lambda} \otimes \chi_{(s-r-2)/2}$, where $\varepsilon = (-1)^{(s-r)/2}$.

If Λ is Δ_{II}^+-dominant, then $\xi \simeq \overline{\pi}_{a_\lambda}^{(1)} \otimes \chi_{(r-s-2)/2}$.

If Λ is Δ_{III}^+-dominant, then $\xi \simeq \overline{\pi}_{a_\lambda}^{(1)} \otimes \chi_{(r-s-4)/2}$.

In the above descriptions, $a_\lambda = (r+3s)/2 = \langle \lambda, 2\alpha_1 + \alpha_2 \rangle$. For the definitions of $\overline{\pi}_k^{(1)}$, r or s see §2.1.

1. Structure of a simple Lie group of type G_2

1.1. Structure of G_2 type Lie algebra

Let $\mathfrak{g}$ be a complex simple Lie algebra of type G_2, $\mathfrak{g}_0$ a normal real form of $\mathfrak{g}$, $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ a Cartan decomposition of $\mathfrak{g}_0$, and θ the corresponding Cartan involution. Here, $\mathfrak{k}_0 = \{X \in \mathfrak{g}_0 \,|\, \theta X = X\}$, $\mathfrak{p}_0 = \{X \in \mathfrak{g}_0 \,|\, \theta X = -X\}$. We denote the complexifications of $\mathfrak{k}_0$, $\mathfrak{p}_0$ etc. by $\mathfrak{k}$, $\mathfrak{p}$ etc., omitting the subscripts. Since $\mathfrak{k}_0 \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, rank $\mathfrak{g}$ = rank $\mathfrak{k} = 2$, and we can take a compact Cartan subalgebra $\mathfrak{t}_0 \subset \mathfrak{k}_0$ of $\mathfrak{g}_0$.

We denote the root system of $\mathfrak{g}$ relative to $\mathfrak{t}$ by Δ, and a positive system of Δ and the Weyl group of Δ are denoted by Δ^+ and W respectively. Then,

$$\Delta = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2), \pm(2\alpha_1+\alpha_2), \pm(3\alpha_1+\alpha_2), \pm(3\alpha_1+2\alpha_2)\}.$$

Here, two roots α_1 and α_2 satisfy the following relations:

$$|\alpha_2|^2 = 3|\alpha_1|^2 = \frac{1}{4}, \quad \langle \alpha_1, \alpha_2^\vee \rangle = -1, \quad \langle \alpha_2, \alpha_1^\vee \rangle = -3,$$

where $\check{\alpha}$ is a coroot for α in Δ. The system of compact (resp. non-compact) roots is denoted by Δ_c (resp. Δ_n) and put $\Delta_c^+ = \Delta^+ \cap \Delta_c$, $\Delta_n^+ = \Delta^+ \cap \Delta_n$. We may assume that $\Delta_c^+ = \{\alpha_1, 3\alpha_1 + 2\alpha_2\}$.

Let $B(\cdot,\cdot)$ be the Killing form of $\mathfrak{g}$ and $\overline{X}$ the complex conjugate of $X \in \mathfrak{g}$ relative to the compact real form $\mathfrak{k}_0 \oplus \sqrt{-1}\mathfrak{p}_0$ of $\mathfrak{g}$. We equip $\mathfrak{g}$ with an inner product $(\cdot,\cdot)$ defined by $(X, Y) = -B(X, \overline{Y})$. Consider the root space decomposition $\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \ (\forall H \in \mathfrak{t})\}$. Then there exists an element E_α of $\mathfrak{g}_\alpha$ for each root α such that

$$B(E_\alpha, E_{-\alpha}) = \frac{2}{|\alpha|^2}, \quad E_{-\alpha} = -\overline{E}_\alpha, \tag{1.1}$$

and we define $H_\alpha \in \mathfrak{t}$ by $H_\alpha = [E_\alpha, E_{-\alpha}]$. Moreover, we can take E_α's in the following way:

$$[E_{10}, E_{01}] = E_{11}, \tag{1.2}$$

$$[E_{10}, E_{11}] = 2E_{21}, \tag{1.3}$$

$$[E_{10}, E_{21}] = 3E_{31}, \tag{1.4}$$

$$[E_{32}, E_{-3,-1}] = E_{01}. \tag{1.5}$$

Here, E_{ij} stands for $E_{i\alpha_1+j\alpha_2}$, and E_{ij}'s are uniquely determined under relations (1.1)–(1.5) above when E_{10} and E_{01} are given.

From now on, E_{ij}'s are assumed to satisfy relations (1.1)–(1.5), and define $\tilde{H}_1$, $\tilde{H}_2$ and $\mathfrak{a}_0$ by

$$\begin{aligned} \tilde{H}_1 &= E_{01} + E_{0,-1}, \\ \tilde{H}_2 &= E_{21} + E_{-2,-1}, \\ \mathfrak{a}_0 &= \mathbb{R}\tilde{H}_1 + \mathbb{R}\tilde{H}_2. \end{aligned}$$

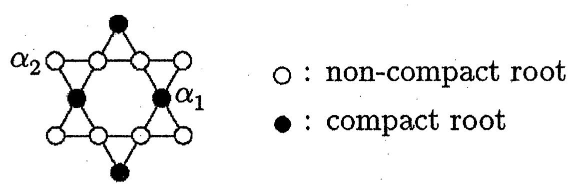

Figure 1: The root system Δ

Then we see that $\mathfrak{a}_0$ is a maximal abelian subspace of $\mathfrak{p}_0$.We equip $\mathfrak{a}_0^*$ with the lexicographic order with respect to the ordered basis $(\tilde{H}_1, \tilde{H}_2)$ of $\mathfrak{a}_0$. We denote the system of restricted roots of $\mathfrak{g}_0$ relative to $\mathfrak{a}_0$ by Ψ and the positive system of Ψ by Ψ^+. Then,

$$\Psi^+ = \{\nu_1,\, \nu_2,\, \nu_1+\nu_2,\, 2\nu_1+\nu_2,\, 3\nu_1+\nu_2,\, 3\nu_1+2\nu_2\}\,.$$

Here, ν_1 and ν_2 are linear forms on $\mathfrak{a}$ defined through the conditions:

$$\nu_1(\tilde{H}_1) = 0,\ \nu_1(\tilde{H}_2) = 2,\ \nu_2(\tilde{H}_1) = 1,\ \nu_2(\tilde{H}_2) = -3\,.$$

Using this Ψ^+, we define a subspace $\mathfrak{n}_0$ of $\mathfrak{g}_0$ by $\mathfrak{n}_0 = \sum_{\lambda\in\Psi^+}(\mathfrak{g}_0)_\lambda$, where $(\mathfrak{g}_0)_\lambda = \{X \in \mathfrak{g}_0 \mid [H, X] = \lambda(H)X\ (\forall H \in \mathfrak{a}_0)\}$. Then we have an Iwasawa decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ of $\mathfrak{g}_0$.

Now define an automorphism u of $\mathfrak{g}$ by

$$u = \left(\exp\frac{\pi}{4}\operatorname{ad}(E_{01} - E_{0,-1})\right)\left(\exp\frac{\pi}{4}\operatorname{ad}(E_{21} - E_{-2,-1})\right)\,.$$

Then u maps $\mathfrak{t}$ onto $\mathfrak{a}$, and two root systems Δ and Ψ are related as $\nu_1\circ u = -(2\alpha_1+\alpha_2)$ and $\nu_2\circ u = 3\alpha_1+\alpha_2$. Using this automorphism u, define a root vector $Z_{ij} \in \mathfrak{g}_{i\nu_1+j\nu_2}$ by

$$\begin{aligned} Z_{10} &= u(E_{-2,-1})\,, & Z_{01} &= u(E_{31})\,,\\ Z_{11} &= u(E_{10})\,, & Z_{21} &= u(E_{-1,-1})\,,\\ Z_{31} &= u(E_{-3,-2})\,, & Z_{32} &= u(E_{0,-1})\,. \end{aligned}$$

Note that Z_{01} and Z_{21} are elements of $\mathfrak{g}_0$ and Z_{10}, Z_{11}, Z_{31} and Z_{32} are in $\sqrt{-1}\mathfrak{g}_0$.

1.2. Structure of the group G and its maximal compact subgroup

Let $G_\mathbb{C}$ be a connected, simply connected simple Lie group with Lie algebra $\mathfrak{g}$, G the analytic subgroup of $G_\mathbb{C}$ with Lie algebra $\mathfrak{g}_0$. The Iwasawa decomposition of G corresponding to that of $\mathfrak{g}_0$ is denoted by $G = KAN$. We know that $K \simeq (SU(2)\times SU(2))/D$ with $D = \{1,\, (-1_2, -1_2)\}$. Here 1_2 is the unit matrix of degree 2. For each element $k \in SU(2)\times SU(2)$, the image of k under the covering homomorphism of $SU(2)\times SU(2)$ onto K is denoted by $k^{\ddagger}$

Put $M=\{m\in K \mid \mathrm{Ad}(m)|_{\mathfrak{a}_0}=\mathrm{id}_{\mathfrak{a}_0}\}$, then we obtain, by a straightforward calculation, that $M=\{1, m_1, m_2, m_1m_2\}$ with m_1 and $m_2\in K$ given by

$$\begin{aligned} m_1 &= \left(\begin{pmatrix}\sqrt{-1} & 0\\ 0 & -\sqrt{-1}\end{pmatrix}, \begin{pmatrix}-\sqrt{-1} & 0\\ 0 & \sqrt{-1}\end{pmatrix}\right)^{\ddagger}, \\ m_2 &= \left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right)^{\ddagger}. \end{aligned}$$

Therefore M is generated by two elements m_1 and m_2 with $m_1^2=m_2^2=1$, $m_1m_2=m_2m_1$, and $M\simeq \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$.

1.3. Structure of maximal parabolic subgroups of G

Here we are going to describe the structure of parabolic subgroups of G. Let Ψ_s be the simple system of Ψ^+, that is, $\Psi_s=\{\nu_1, \nu_2\}$. Then, for each maximal proper subset S of Ψ^+, there is a corresponding maximal parabolic subgroup P_S and any maximal parabolic subgroup of G is conjugate to one of these P_S's. For simplicity, we denote $P_{\{\nu_1\}}$ and $P_{\{\nu_2\}}$ by P_1 and P_2 respectively. The Langlangs decomposition of P_j is denoted by $P_j=M_jA_jN_j$ and let $\mathfrak{m}_j$, $\mathfrak{a}_j$ and $\mathfrak{n}_j$ be Lie algebras of M_j, A_j and N_j respectively. The identity component of M_j is written by $(M_j)_0$.

· **Structure of P_1**

The Lie algebras $\mathfrak{a}_1$, $\mathfrak{m}_1$ and $\mathfrak{n}_1$ are given by

$$\begin{aligned} \mathfrak{a}_1 &= \mathbb{R}\tilde{H}_1\,, \\ \mathfrak{m}_1 &= (\mathfrak{g}_0)_{\nu_1}\oplus\mathbb{R}\tilde{H}_2\oplus(\mathfrak{g}_0)_{-\nu_1}\,, \\ \mathfrak{n}_1 &= (\mathfrak{g}_0)_{\nu_2}\oplus(\mathfrak{g}_0)_{\nu_1+\nu_2}\oplus(\mathfrak{g}_0)_{2\nu_1+\nu_2}\oplus(\mathfrak{g}_0)_{3\nu_1+\nu_2}\oplus(\mathfrak{g}_0)_{3\nu_1+2\nu_2}\,. \end{aligned}$$

The subalgebra $\mathfrak{m}_1$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and and we have $(\sqrt{-1}Z_{10}, \tilde{H}_2, -\sqrt{-1}\theta Z_{10})$ as its $\mathfrak{sl}_2$-triplet. Put $F_1=\{1, m_2\}\subset M$, then $M_1=F_1(M_1)_0$ and the action of $m_2\in F_1$ on $(M_1)_0$ is as follows:

$$(M_1)_0\simeq SL(2,\mathbb{R})\ni x\longmapsto JxJ^{-1}\in SL(2,\mathbb{R})\,.$$

Here, $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and we identify M_{10} with $SL(2,\mathbb{R})$ through the identification of the above $\mathfrak{sl}_2$-triplet with the canonical $\mathfrak{sl}_2$-triplet $\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)$ of $\mathfrak{sl}(2,\mathbb{R})$.

· **Structure of P_2**

The Lie algebras $\mathfrak{m}_2$, $\mathfrak{a}_2$ and $\mathfrak{n}_2$ are given by

$$\begin{aligned}
\mathfrak{a}_2 &= \mathbb{R}(3\tilde{H}_1 + \tilde{H}_2)\,, \\
\mathfrak{m}_2 &= (\mathfrak{g}_0)_{\nu_2} \oplus \mathbb{R}(\tilde{H}_1 - \tilde{H}_2) \oplus (\mathfrak{g}_0)_{-\nu_2}\,, \\
\mathfrak{n}_2 &= (\mathfrak{g}_0)_{\nu_1} \oplus (\mathfrak{g}_0)_{\nu_1+\nu_2} \oplus (\mathfrak{g}_0)_{2\nu_1+\nu_2} \oplus (\mathfrak{g}_0)_{3\nu_1+\nu_2} \oplus (\mathfrak{g}_0)_{3\nu_1+2\nu_2}\,.
\end{aligned}$$

The subalgebra $\mathfrak{m}_2$ is also isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and we can take $(Z_{01}, (\tilde{H}_1 - \tilde{H}_2)/2, -\theta Z_{01})$ as its $\mathfrak{sl}_2$-triplet. Put $F_2 = \{1, m_1\} \subset M$, then $M_2 = F_2(M_2)_0$ and M_2 is isomorphic to M_1.

2. Discrete series of G and their embeddings

2.1. Finite-dimensional irreducible K-modules and M_j-modules

· **Irreducible K-modules**

Let λ be a Δ_c^+-dominant, integral linear form on $\mathfrak{t}$ and $(\tau_\lambda, V_\lambda)$ a finite-dimensional irreducible representation of $\mathfrak{k}$ with highest weight λ. Then, as $\mathfrak{k}$-modules,

$$\begin{aligned}
V_\lambda &\simeq V_r \hat{\otimes} V_s \quad \text{for} \quad r = \lambda(H_{10}),\ s = \lambda(H_{32}), \\
\mathfrak{p} &\simeq (V_3 \hat{\otimes} V_1)\,.
\end{aligned}$$

Here, V_d is the $(d+1)$-dimensional irreducible $SU(2)$-module. Let $\{e_p^{(d)} \mid p = -d, -d+2, \dots, d-2, d\}$ be an orthonormal basis of V_d consisting of weight vectors, where $e_p^{(d)}$ is a weight vector for weight p. Then we have an orthonormal basis $\{e_p^{(r)} \otimes e_q^{(s)} \mid p = -r, -r+2, \dots, r; q = -s, -s+2, \dots, s\}$ for $V_r \otimes V_q$. For simplicity, we write $e_{pq}^{(rs)}$ for $e_p^{(r)} \otimes e_q^{(s)}$.

· **Irreducible M_j-modules**

Next, we prepare some notation for irreducible M_j-modules. Two groups M_1 and M_2 are isomorphic to $F \ltimes SL(2,\mathbb{R})$, where $F \simeq \mathbb{Z}/2\mathbb{Z}$. Here we identify F with the subgroup F_1 or F_2 given in the previous subsection. If we denote the generator of F by a, then for an element g of $SL(2,\mathbb{R})$, $aga^{-1} = JgJ^{-1} \in SL(2,\mathbb{R})$, with $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

For each non-negative integer n, define a linear isomorphism ψ_n of V_n onto itself by

$$\psi_n(e_p^{(n)}) = (-1)^{(n-p)/2} e_p^{(n)},$$

for $p = -n, -n+2, \ldots, n$. Using this isomorphism ψ_n, we define two $(n+1)$-dimensional representations $\sigma_{n,1}^{(1)}$ and $\sigma_{n,-1}^{(1)}$ of $F \ltimes SL(2,\mathbb{R})$ as follows.

Representation spaces of $\sigma_{n,1}^{(1)}$ and of $\sigma_{n,-1}^{(1)}$ are both V_n. Action of $SL(2,\mathbb{R})$ on V_n is denoted by π_n and for $g \in SL(2,\mathbb{R})$, put

$$\sigma_{n,1}^{(1)}(g) = \sigma_{n,-1}^{(1)}(g) = \pi_n(g).$$

For the action of F, $\sigma_{n,1}^{(1)}(a)$ and $\sigma_{n,-1}^{(1)}(a)$ are defined by

$$\sigma_{n,\pm 1}^{(1)}(a) = \pm\psi_n.$$

Then these two representations give finite-dimensional irreducible representations of $F \ltimes SL(2,\mathbb{R}) \simeq M_j$ $(j = 1,2)$.

In the following discussion, we denote the representation space for the irreducible representation $\sigma_{n,\pm 1}^{(1)}$ by $V_{n,\pm 1}$.

For an integer n, $|n| > 1$, let $\pi_n^{(1)}$ be the discrete series representation of $SL(2,\mathbb{R})$ having n as its highest weight or lowest weight, and H_n the representation space of $\pi_n^{(1)}$. We can take an orthonormal basis $\{v_p^{(n)}\}$ of H_n consisting of weight vectors, where $v_p^{(n)}$ is a weight vector for weight p and $p = n, n+2, \ldots$ (if $n > 0$), $p = n, n-2, \ldots$ (if $n < 0$). Then, there is a linear isomorphism T of H_n onto H_{-n} such that T maps $v_p^{(n)}$ to $v_{-p}^{(-n)}$. Now, we introduce another irreducible representation $\overline{\pi}_n$ of $F \ltimes SL(2,\mathbb{R})$. As an $SL(2,\mathbb{R})$-module, $\overline{\pi}_n = \pi_n^{(1)} \oplus \pi_{-n}^{(1)}$ and the action of $a \in F$ is given by $\overline{\pi}_n(a)(v_p^{(n)}, v_{-q}^{(-n)}) = (T^{-1}v_{-q}^{(-n)}, Tv_p^{(n)}) = (v_q^{(n)}, v_{-p}^{(-n)})$.

2.2. Parametrization of discrete series representations of G

Let Ξ_c be the totality of Δ_c^+-dominant, regular, integral linear forms Λ on $\mathfrak{t}$. For each $\Lambda \in \Xi_c$, take a positive system $\Delta^+ = \Delta^+(J)$ of Δ in such a way that Λ is Δ^+-dominant. We can parametrize the discrete series representations of G by Ξ_c, denoting the discrete series of G with Harish-Chandra parameter Λ by π_Λ.

Let Δ_J^+ $(J = I, II, III)$ be positive systems of Δ defined as follows:

$$\begin{aligned}
\Delta_I^+ &= \{\alpha_1,\, \alpha_2,\, \alpha_1+\alpha_2,\, 2\alpha_1+\alpha_2,\, 3\alpha_1+\alpha_2,\, 3\alpha_1+2\alpha_2\}, \\
\Delta_{II}^+ &= \{\alpha_1+\alpha_2,\, -\alpha_2,\, \alpha_1,\, 2\alpha_1+\alpha_2,\, 3\alpha_1+2\alpha_2,\, 3\alpha_1+\alpha_2\}, \\
\Delta_{III}^+ &= \{-\alpha_1-\alpha_2,\, 3\alpha_1+2\alpha_2,\, 2\alpha_1+\alpha_2,\, \alpha_1,\, -\alpha_2,\, 3\alpha_1+\alpha_2\}.
\end{aligned}$$

For a discrete series π_Λ of G, we may assume that the corresponding positive system $\{\alpha \in \Delta \mid (\alpha, \Lambda) > 0\} \subset \Delta$ is one of the above Δ_J^+'s. Define three subsets Ξ_J $(J = I, II, III)$ of Ξ_c by $\Xi_J = \{\Lambda \in \Xi_c \mid \Delta^+ = \Delta_J^+\}$. Put $\rho_c = \frac{1}{2}\sum_{\alpha\in\Delta_c^+} \alpha$, $\rho_n = \frac{1}{2}\sum_{\alpha\in\Delta_n^+} \alpha$ and $\lambda = \Lambda - \rho_c + \rho_n$. This linear form λ is called the *Blattner parameter* of π_Λ and the discrete series π_Λ has the lowest K-type τ_λ.

2.3. Method for the determination of embeddings

To determine embeddings of discrete series of G into its generalized principal series, we use the same method as in [7, 9].

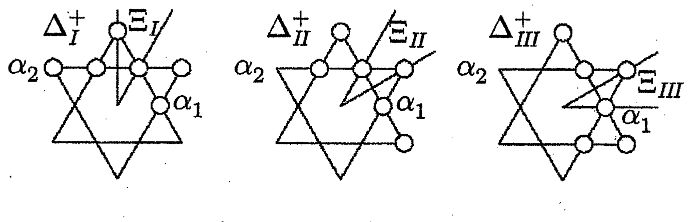

Figure 2: Three possible positive systems

For a finite-dimensional representation (τ, V) of K, we introduce a function space

$$C^\infty_\tau(G) = \{f : G \xrightarrow{C^\infty} V \mid f(kg) = \tau(k)f(g)\ (\forall k \in K, g \in G)\}.$$

Since K acts on $\mathfrak{p}$ by adjoint action, we can take a tensor product representation $\tau_\lambda \otimes \mathrm{Ad}|_\mathfrak{p}$. This tensor product is decomposed as $\tau_\lambda \otimes \mathrm{Ad}|_\mathfrak{p} \simeq \tau^+ \oplus \tau^-$, where τ^+ (resp. τ^-) is the sum of irreducible components of $\tau_\lambda \otimes \mathrm{Ad}|_\mathfrak{p}$ with height weight of the form $\lambda + \alpha$, $\alpha \in \Delta_n^+$ (resp. $-\alpha \in \Delta_n^+$). According to this decomposition, the space $V_\lambda \otimes \mathfrak{p}$ is decomposed as $V_\lambda \otimes \mathfrak{p} \simeq V^+ \oplus V^-$, where $V^\pm$ is the sum of K-submodules corresponding to $\tau^\pm$. Then we have a projection P_λ of $V_\lambda \otimes \mathfrak{p}$ onto V^- along this decomposition.

Now we define the main tool of our method, differential operator $\mathcal{D}_\lambda$, as follows: take an orthonormal basis $\{X_j\}$ of $\mathfrak{p}$ with respect to the inner product $(\cdot,\cdot)$ and for functions f in $C^\infty_{\tau_\lambda}(G)$, put

$$(\nabla f)(g) = \sum_j (L_{X_j} f)(g) \otimes \overline{X}_j ,$$
$$(\mathcal{D}_\lambda f)(g) = P_\lambda(\nabla f)(g) ,$$

where L_{X_j} is the differentiation by X_j as a left invariant vector field. For the explicit description of the operator $\mathcal{D}_\lambda$, see [9, §§3.3–3.5].

Take a parabolic subgroup P of G, and let $P = M_P A_P N_P$ be its Langlands decomposition. For an irreducible admissible representation σ of M_P and a linear form μ on $\mathfrak{a}_P$, $\xi = \sigma \otimes e^\mu$ is an irreducible admissible representation of the Levi part $M_P A_P$. Put $\tilde{\xi} = \sigma \otimes e^{\mu + \rho_P}$. Here, $\rho_P(H) = \frac{1}{2}\operatorname{tr}(\operatorname{ad} H|_{\mathfrak{n}_P})$ $(H \in \mathfrak{a}_P)$. For a character η of N_P, put $\mathcal{D}_{\lambda,\eta}$ be the restriction of $\mathcal{D}_\lambda$ to the subspace

$$C^\infty_{\tau_\lambda}(G, \eta) = \{f \in C^\infty_{\tau_\lambda}(G) \mid f(gn) = \eta(n)^{-1} f(g)\ (\forall g \in G, n \in N_P)\}.$$

We also write $\mathcal{D}_{\lambda, 1_{N_{P_1}}}$ by $\mathcal{D}_{\lambda, 1_{N_1}}$ for simplicity. Then we have the following facts.

Theorem 2.1 (cf. [7, Theorem 2.4]). *There is a linear isomorphism*

$$\mathrm{Hom}_{(\mathfrak{g},K)}(\pi^*_\Lambda, \mathrm{Ind}^G_{N_P}(\eta)) \simeq \operatorname{Ker} \mathcal{D}_{\lambda,\eta} .$$

Theorem 2.2 (cf. [7, Theorem 3.5]). *There is a linear isomorphism*

$$\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda^*, \mathrm{Ind}_P^G(\xi \otimes 1_{N_P})) \simeq \mathrm{Hom}_{(\mathfrak{l},K_L)}(\tilde{\xi}^*, \mathrm{Ker}\,\mathcal{D}_{\lambda,1_{N_P}}).$$

In these two theorems, π_Λ^* denotes the discrete series of G contragredient to π_Λ. Note that the case of groups of type G_2, every discrete series representation is self-contragredient, that is, $\pi_\Lambda^* \simeq \pi_\Lambda$.

Remark.

(1) Theorem 2.4 in [7] is proved much wider class of representation η, but we need the results only for characters.

(2) In [7], Theorems 2.1, 2.2 are proved under the restriction that the Blattner parameter λ is "far from the walls", but this condition is no longer necessary.

2.4. Description of the embeddings

The theorems in the previous subsection say that if we solve the differential equation $\mathcal{D}_{\lambda,\eta} f = 0$ (and determine its fine structure) then we can obtain the embeddings of discrete series. For example, by finding out the M_1A_1-module structure of $\mathrm{Ker}\,\mathcal{D}_{\lambda,1_{N_1}}$, we have the following result.

Theorem 2.3. *Discrete series π_Λ can be embedded into $\mathrm{Ind}_{P_1}^G(\xi \otimes 1_{N_1})$ with the representations ξ of M_1A_1 listed below with multiplicity 1.*

- *If Λ is Δ_I^+-dominant, then $\xi = \sigma_{r,\varepsilon}^{(1)} \otimes \chi_{s-1}$, $\overline{\pi}_{a_\lambda} \otimes \chi_{-(s-r-2)/2}$, where $\varepsilon = (-1)^{(s+r)/2}$.*

- *If Λ is Δ_{II}^+-dominant, then $\xi = \overline{\pi}_{a_\lambda} \otimes \chi_{(r-s-2)/2}$.*

- *If Λ is Δ_{III}^+-dominant, then $\xi = \overline{\pi}_{a_\lambda} \otimes \chi_{(r-s-4)/2}$.*

Here, $a_\lambda = (r+3s)/2$ and χ_a, $a \in \mathbb{R}$, is the character of A defined by $\chi_a(\exp t\tilde{H}_2) = \exp(at)$.

Since the proof of this result, or the process of solving the differential equation, is too long and elaborate, we omit it here and the complete proof will be published later.

Remark. In the proof of this result, we solve the equation $\mathcal{D}_{\lambda,1_{N_1}}$ under some condition in λ stronger than the condition "far from the walls". But by the aid of the translation functor introduced in [10], we can observe that the result 2.3 is valid for every regular Λ.

3. Partial results and remaining problems

In the preceding argument, we considered the operator $\mathcal{D}_{\lambda,\eta}$ with the trivial character of the unipotent radical of a parabolic subgroup. If we change the character η to arbitrary one, then we can find out the generalized Whittaker models of discrete series. The author tried to determine $\operatorname{Ker}\mathcal{D}_{\lambda,\eta}$ for (may be degenerate) character of the unipotent radical of a minimal parabolic subgroup and there is a partial result.

In the following, we assume that the Harish-Chandra parameter Λ of π_Λ is Δ_I^+-dominant. Define linear forms μ_j $(j=1,2)$ on $\mathfrak{a}_0$ by

$$\begin{aligned} \mu_1(\tilde{H}_1) &= -(s+2)\,, & \mu_1(\tilde{H}_2) &= r\,, \\ \mu_2(\tilde{H}_1) &= -\tfrac{1}{2}(s-r+4)\,, & \mu_2(\tilde{H}_2) &= -\tfrac{1}{2}(r+3s)\,. \end{aligned}$$

Let ψ_j $(j=1,2)$ be analytic functions defined by

$$\psi_1(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} t^n\,,$$
$$\psi_2(t) = \sum_{n=0}^{\infty} 2\Big(\sum_{m=1}^{n} \frac{1}{m}\Big)\frac{(-1)^n}{(n!)^2} t^n\,.$$

Note that $\psi_1(x^2/4)$ is the Bessel function $J_0(x)$. Using these ψ_j's, we define two functions φ_1, φ_2 on A as follows:

$$\varphi_1(a) = a^{\mu_1}\exp(\eta_1 a^{-\nu_1})\psi_1(\tfrac{1}{4}\eta_2^2 a^{-2\nu_2})\,,$$
$$\varphi_2(a) = a^{\mu_1}\exp(\eta_1 a^{-\nu_1})\{\psi_2(\tfrac{1}{4}\eta_2^2 a^{-2\nu_2}) + 2\nu_2(\log a)\psi_1(\tfrac{1}{4}\eta_2^2 a^{-2\nu_2})\}\,.$$

Here, for a linear form μ on $\mathfrak{a}_0$ and $a\in A$, $a^\mu = \exp(\mu(\log a))$, and $\eta_j = \eta(\tilde{H}_j)$.

Now we introduce five V_λ-valued functions f_* $(* = 0, +, -, 1, 2)$ on A. For f_0, f_+ and f_-, they are defined by

$$f_0(a) = \sum_p \gamma_{p,-p} a^{\mu_1} e^{(rs)}_{p,-p}\,,$$
$$f_+(a) = a^{\mu_2} \exp(\eta_1 a^{-\nu_1}) e^{(rs)}_{rs}\,,$$
$$f_-(a) = a^{\mu_2} \exp(-\eta_1 a^{-\nu_1}) e^{(rs)}_{-r,-s}\,,$$

for $a \in A$. Here the sum is taken for $p = -r, -r+2, \ldots, r$, and

$$\gamma_{pq} = \sqrt{\binom{r}{\frac{1}{2}(r-p)}\binom{\frac{1}{2}(s+r)}{\frac{1}{2}(s-q)}\binom{\frac{1}{2}(s+q)}{\frac{1}{2}(s-r)}^{-1}}\,.$$

Two remaining functions f_1 and f_2 are given by

$$f_j(a) = \sum_{p,q} \gamma_{pq} (\eta_2^{-1} a^{\nu_2})^{|p+q|/2} c^{(j)}_{pq}(a) e^{(rs)}_{pq} \quad (j = 1, 2)\,,$$

where

$$c^{(j)}_{pq} =$$
$$\begin{cases} \varphi_j & \text{if } p+q=0 \\ (L_{\tilde{H}_1} - s - p - q) \cdots (L_{\tilde{H}_1} - s - 4)(L_{\tilde{H}_1} - s - 2)\varphi_j & \text{if } p+q>0 \\ (-1)^{(p+q)/2}(L_{\tilde{H}_1} - s + p + q) \cdots (L_{\tilde{H}_1} - s - 4)(L_{\tilde{H}_1} - s - 2)\varphi_j & \text{if } p+q<0\,. \end{cases}$$

Extend these f_*'s to G by $f_*(kan) = \eta(n)^{-1}\tau_\lambda(k) f_*(a)$ for $k \in K$, $a \in A$ and $n \in N$. Then f_*'s are functions in $C^\infty_{\tau_\lambda}(G;\eta)$ and the following lemma describes $\operatorname{Ker}\mathcal{D}_{\lambda,\eta}$.

Lemma 3.1. *If* Λ *is* Δ_I^+*-dominant, then the dimension of* $\operatorname{Ker}\mathcal{D}_{\lambda,\eta}$ *is* (i) *zero,* (ii) *two, and* (iii) *three, according to the cases* (i) $\eta_1 \neq 0$ *and* $\eta_2 \neq 0$, (ii) $\eta_1 \neq 0$, $\eta_2 = 0$ *or* $\eta_1 = 0$, $\eta_2 \neq 0$, *and* (iii) $\eta_1 = \eta_2 = 0$. *In cases* (ii) *and* (iii), *a basis of* $\operatorname{Ker}\mathcal{D}_{\lambda,\eta}$ *is given as follows:*

$$\begin{array}{lll} \{f_+,\, f_-\} & \text{if} & \eta_1 \neq 0,\, \eta_2 = 0\,, \\ \{f_1,\, f_2\} & \text{if} & \eta_1 = 0,\, \eta_2 \neq 0\,, \\ \{f_0,\, f_+,\, f_-\} & \text{if} & \eta_1 = \eta_2 = 0\,. \end{array}$$

If Λ is Δ_I^+-dominant, then $\mathrm{Dim}\,\pi_\Lambda = 5 < 6 = \dim \mathfrak{n}$, where Dim stands for the Gelfand-Kirillov dimension. So, general theory tells us that $\mathrm{Ker}\,\mathcal{D}_{\lambda,\eta} = \{0\}$ for non-degenerate η with Δ_I^+-dominant Λ. Note that, for our group, π_Λ has Whittaker models if and only if Λ is Δ_{II}^+-dominant.

Therefore the most interesting case is the case of Δ_{II}^+-dominant Λ, but recently the case of Δ_I^+ also causes our interest in connection with Borel-de Siebanthal discrete series. The previous lemma may be a trifle, but it could be a basepoint for attacking more interesting case, that is, generalized Whittacker models.

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